Mean Value Theorem

1) Rolle's Theorem
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ROLLE'S THEOREM

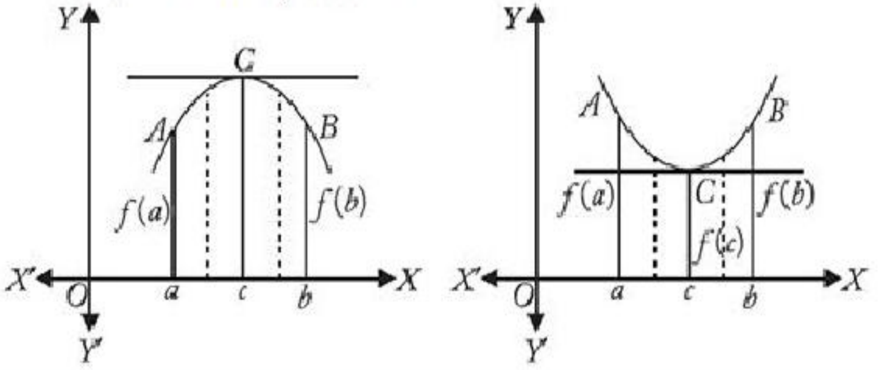
MEAN VALUE THEOREM

$f : [a,b] \longrightarrow R$

Rolle's Theorem

- If a real valued function $f(x)$
  - (i) is continuous in $[a, b]$
  - (ii) is differentiable in (a, b)
  - (iii) $f(a) = f(b)$,

  then there exist at least one real number c in the interval (a, b) such that $f'(c) = 0$.

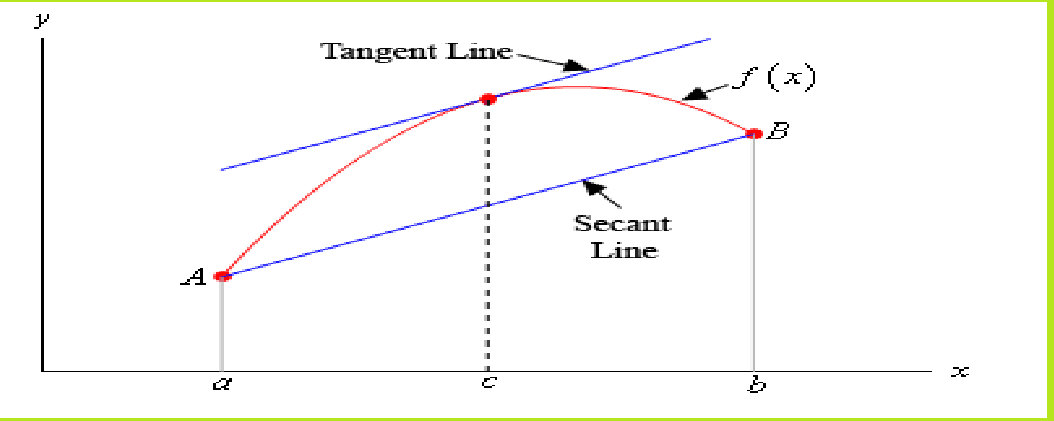


Mean Value Theorem

$f : [a,b] \longrightarrow R$

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Verify Mean Value Theorem for

$f(x) = x^2 - 4x + 3$ in $[1, 4]$

The given function is $f(x) = x^2 - 4x + 3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$f(1) = 1^2 - 4\times 1 + 3 = 0$, $f(4) = 4^2 - 4\times 4 + 3 = 3$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{3 - (0)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$$f'(c) = 1$$
$$\Rightarrow 2c - 4 = 1$$
$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)$$

Hence, Mean Value Theorem is verified for the given function

VERIFY ROLLE'S THEOREM FOR

Verify rolle's theorem for each of the following functions on the indicated intervals:
$f(x) = \cos 2x$ on $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

$$f(x) = \cos 2x \text{ on } \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$$

We know that $\cos x$ is a continuous and differentiable every where. So,
$f(x)$ is continuous in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$ and differentiable is $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$.

Now,
$$f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$
$$f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$
$$\Rightarrow \quad f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$
such that $f'(c) = 0$

Now,
$$f'(x) = 2 \sin 2x$$
$$f'(c) = 2 \sin 2c = 0$$
$$\Rightarrow \quad \sin 2c = 0$$
$$\Rightarrow \quad 2c = 0$$
$$\Rightarrow \quad c = 0 \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$$

Hence Verified.

Verify Lagrange's mean value theorem for the following function on the indicated interval. In each case find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem

$f(x) = x^2 - 1$ on $[2, 3]$

Here,

$$f(x) = x^2 - 1 \text{ on } [2,3]$$

It is a polynomial function so it is continuous in $[2, 3]$ and differentiable in $(2,3)$. So, both conditions of Lagrange's mean value theorem are satisfied.

Therefore, there exist a point $c \in (2,3)$ such that

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$2c = \frac{\left((3)^2 - 1\right) - \left((2)^2 - 1\right)}{1}$$

$$2c = (8 - 3)$$

$$c = \frac{5}{2} \in (2,3)$$

Hence, Lagrange's mean value theorem is verified.

CHECK IF ROLLE'S THEOREM IS APPLICABLE OR NOT

$$f(x) = 8 + |x - 3|.$$

Let $f(x) = 8 + |x - 3|$. on [1,5]

Since Modulus function is continuous, f(x) is continuous

$$f(1) = 8 + |1 - 3| = 8 + |-2| = 8 + 2 = 10$$

$$f(5) = 8 + |5 - 3| = 8 + |2| = 8 + 2 = 10$$

$$f'(x) = \begin{cases} -1, & \text{if } x < 3 \\ 1, & \text{if } x > 3 \end{cases}$$

NOT APPLICABLE

f is not differentiable at 3,

so f is not differentiable on $(1, 5)$.

X

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals

$f(x) = 3 + (x-2)^{\frac{2}{3}}$ on $[1,3]$.

$f(x) = 3 + (x-2)^{\frac{2}{3}}$ on $[1,3]$

Differentiating it with respect to x,

$$f'(x) = \frac{2}{3} \times \frac{1}{(x-2)^{\frac{1}{3}}}$$

Thus, $f(x)$ is not differentiable at $x = 2 \in (1,3)$

Hene, Rolle's theorem is not applicable for $f(x)$ in $x \in [1,3]$.

Find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining $(3, 0)$ and $(4, 1)$.

Here,

$$y = (x-3)^2$$

Since, y is a polynomial function, so it continuous differentiable,

$\Rightarrow$ Lagrange's mean value theorem is applicable

$\Rightarrow$ There exist a point c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 2(c-3) = \frac{f(4) - f(3)}{4 - 3}$$

$$\Rightarrow \quad 2c - 6 = \frac{1 - 0}{1}$$

$$\Rightarrow \quad 2c = 7$$

$$\Rightarrow \quad c = \frac{7}{2}$$

$$\Rightarrow \quad y = \left(\frac{7}{2} - 3\right)^2 \quad \Rightarrow \quad y = \frac{1}{4}$$

So, $(c, y) = \left(\frac{7}{2}, \frac{1}{4}\right)$ is the required point.